Can we implement the holographic principle in asymptotically flat spacetimes?

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February 1, 2008

Abstract

We discuss some recent results in the quest to implement the holographic principle in asymptotically flat spacetimes. In particular we introduce the key ingredients of the candidate dual theory which lives at null infinity and it is invariant under the asymptotic symmetry group of this class of spacetimes.

1 Introduction

The last few decades witnessed the rise of several and often antithetical methods aimed at the quantization of general relativity. Unfortunately, up to now, none of them has been recognized as fully satisfactory and the quest to derive a quantum version of Einstein theory still rages on. Beside the non renormalizability of Einstein-Hilbert Lagrangian in a perturbative scheme, one of the main obstruction to the success of the above programme lies in the existence of peculiar objects such as black holes which are the source of extreme gravitational fields. Already at a classical level, their behaviour drastically differs from the usual physical systems commonly studied at low energies; to support such assertion, one simply needs to recognize that it is possible to associate to a black hole - considered as a dynamical system - three "evolution" laws which are directly intertwined with the laws of thermodynamics (see [1] and references therein for a detailed analysis). In particular, to a black hole, one associates an entropy function related to its geometrical data by means of the widely-known Bekenstein formula[1]

$$S = \frac{A}{4},\tag{1}$$

where A is the area of the event horizon. This relation shows a drastic departure from the behaviour of any other classical systems where entropy is proportional to the volume of the considered region of spacetime. Starting from these premises, 't Hooft

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[1]Unless stated otherwise, in this paper we assume $c = \hbar = G = 1$

remarked that a further striking consequence of (1) arises if one considers S as a (indirect) measure of the number of degrees of freedom accessible to a physical theory formulated inside the black hole itself; the key suggestion is that the proportionality of S with the area could be interpreted as a signal that all the phsycal information stored inside the black hole could be encoded by means of a suitable second theory intrinsically formulated on the event horizon and with a density of data not exceeding the Planck density [2]. This conjecture goes under the name of *holographic principle* and, in the last decade, it has been the driving principle behind several new achievements in quantum field theory.

For our purposes it is imperative to mention only two key steps which have improved the original 't Hooft proposal; the first goes under the name of "covariant entropy conjecture" which generalizes a Bekenstein-like formula to a wider class of spacetime regions [3]. In detail, consider in a 4-dimensional Lorentzian manifold (M, g_{ab}) a spacelike 2-surface Σ and the light rays originating from Σ whose congruences form several light-sheets. To each of these sheets, it is possible to associate the Raychauduri equation which governs the law of evolution for the area A' of $\partial\Sigma$ along the congruences; the conjecture proposed by Bousso states that, whenever A' is monotonically non-increasing and the sheet L_Σ under consideration terminates on the boundary of the spacetime (or on a caustic), the entropy $S(L)$ of the gravitational and the matter fields evolving inside this region of the manifold is bounded by $S(L) \leq \frac{A'}{4}$.

Despite the above conjecture has been demonstrated under some specific and peculiar conditions (see [4] and references therein), an almost straightforward consequence of Bousso conjecture is the chance to extend the holographic principle to any region where the above bound holds: all the information of a theory living inside L can be encoded on the boundary of the sheet with a density of date not exceeding the Planck density. Furthermore the above argument can be reversed and it is possible to ask ourselves if, in a generic but fixed manifold (usually called the "bulk"), it exists a codimension 1 submanifold, also called "screen" (usually but not necessary the boundary), which plays the role of the light sheet L i.e. all the information of a theory living in the whole spacetime can be encoded on it. This is a key consequence of the covariant entropy conjecture since it allows us to identify, by means only of a geometrical and classical construction, certain hypersurfaces in the spacetime as natural candidates where to construct the "holographic theory".

Nonetheless 't Hooft original conjecture lacks any explicit mean to implement the holographic principle even when it is identified a screen where to encode the bulk data. Thus one has to look for a way to formulate the theory on the screen itself case-by-case. A concrete realization of this paradigm has been proposed a few years ago and it goes under the name of AdS/CFT correspondence [5] which states the existence of a one to one correspondence between a superstring theory of type IIB living on $AdS_5 \times S^5$ and a $SU(N)$ super Yang-Mills field theory living on the boundary of AdS_5. Although such conjecture is widely accepted and successfully tested within different context, it strictly requires that an (asymptotically) AdS boundary condition is imposed both on the underlying manifold and on the physical fields. It is thus natural to ask ourselves whether such a conjecture and, in general, the holographic principle can be implemented whenever we consider a different class of spacetimes. In particular, in

2

this paper we discuss some recent results on the scenario with a vanishing cosmological constant.

2 Asymptotically flat spacetimes and the BMS group

Starting from the premises discussed in the previous section, the first step to implement the holographic principle in an asymptotically flat spacetime consists on choosing a suitable codimension 1 submanifold where to encode the data from the bulk theory. According to the covariant entropy conjecture [3], the most natural candidate is future or past null infinity and, due to its importance, we now review the key details of its construction though we refer to [1] and section 2 in [6] for a more exhaustive analysis.

A manifold $(\hat{M}, \hat{g}_{ab})$ is called asymptotically flat at null infinity if it exists a second manifold (M, g_{ab}), a positive scalar function Ω over M (usually called "compactification factor") and an embedding i of $\hat{M}$ in M such that

- $\Im^+ \cup i_0 \cup \Im^- = \partial i(\hat{M})$ where $\Im^{\pm}$ are respectively future and null infinity and where i_0 is spatial infinity,

- $\hat{g}_{ab} = \Omega^2 g_{ab}$ on $i(\hat{M})$, Ω being everywhere smooth except at most i_0 where it is at least twice differentiable,

- $\Omega = 0$, but $d\Omega \neq 0$ point wisely on $\Im^{\pm}$.

According to the above construction both $\Im^+$ and $\Im^-$ are differentiable null manifolds topologically equivalent to $\mathbb{R} \times S^2$; consequently, given a fixed asymptotically flat spacetime $(\hat{M}, \hat{g}_{ab})$ and a fixed compactification factor Ω, its boundary structure at (future or past) null infinity is *intrinsically* characterized by the following data: $\Im^{\pm}$, the restriction of the metric $h_{ab} = g_{ab}|_{\Im^{\pm}}$ and the vector $n^a = g^{ab}\nabla_b\Omega$. Furthermore this triple is also *universal* i.e., given any two asymptotically flat manifolds, say $(\hat{M}_1, \hat{g}_{1ab})$ and $(\hat{M}_2, \hat{g}_{2ab})$, let us associate to each of them an arbitrary triple defining the (future) null boundary structure, say $(\Im_1^+, h_{1ab}, n_2^a)$ and $(\Im_2^+, h_{2ab}, n_2^a)$. Then it always exists a diffeomorphism

$$\gamma : \Im_1^+ \to \Im_2^+ \tag{2}$$

such that $\gamma^* h_{2ab} = h_{1ab}$ and $n_2^a = \gamma_* n_1^a$ which implies that the boundary structure is the same independently from the chosen bulk manifold and, more important, from the specific field theory living on it. For this reason, $\Im^+$ and $\Im^-$ do represent the natural and more general framework where to construct an holographic theory for an arbitrary asymptotically flat spacetime.

The set of all maps, defined as in (2), constitutes the diffeomorphism group of the (future or past) boundary data and its explicit representation can be provided once a suitable coordinate frame on $\Im$ is chosen. The answer to this query goes under the name of "Bondi frame" $(u, \Omega, \theta, \varphi)$ where, besides the natural (θ, φ)-coordinates over S^2 and the conformal factor Ω, we have introduced u, the affine parameter along the null geodesics over $\Im$, i.e. the integral curves of the vector n^a which turns out to be complete. This specific choice of coordinates can be restricted to $\Im$ simply remembering that, by construction, $\Omega|_{\Im} = 0$ and, thus, the above set of diffeomorphisms

becomes the following set of transformations ($z = ctg\theta e^{i\varphi}$):

$$u \longrightarrow u' = K_\Lambda(z, \bar{z}) \left[u + \alpha(z, \bar{z}) \right], \tag{3}$$

$$z \longrightarrow z' = \frac{az + b}{cz + d}, \tag{4}$$

$$\bar{z} \longrightarrow \bar{z}' = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}, \tag{5}$$

where $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$, $\alpha(z, \bar{z})$ is a generic scalar function over S^2 and

$$K_\Lambda = \frac{1 + |z|^2}{|az + b|^2 + |cz + d|^2}.$$

The relations (3), (4) and (5) form the so-called **Bondi-Metzner-Sachs group** (BMS group) and a direct inspection shows that it is the semidirect product between $SL(2, \mathbb{C})$, the universal cover of the proper ortochronous Lorentz group, and N, the set of scalar functions over S^2 (considered as a group under addition). Furthermore there is an arbitrariness in the choice of the topology of N, namely we have the freedom to impose a suitable regularity condition over each $\alpha : S^2 \to \mathbb{R}$ i.e. α could lie in $C^\infty(S^2)$ (nuclear topology) or in $L^2(S^2)$ (Hilbert topology). There is no general mathematical reason to prefer one of the two choices though, in a recent analysis [6], it appears that, from the point of view of an holographic correspondence, the nuclear topology is the unique one which allows to coherently project bulk data to $\Im$. Thus, we can conclude that the diffeomorphism group of the intrinsic and universal boundary structure of any four dimensional asymptotically flat spacetime[2] is

$$BMS = SL(2, \mathbb{C}) \ltimes C^\infty(S^2),$$

where $\ltimes$ stands for "semidirect product" and $C^\infty(S^2)$ is an abelian group usually called "supertranslations".

3 Kinematical and dynamical data for a field theory on null infinity

The identification of $\Im$ as a candidate screen where to encode bulk data in an asymptotically flat spacetime leads to the question whether it is possible to coherently define a (quantum) field theory living only on $\Im$ itself. By the light of our previous discussion, it is natural to require the invariance of such a theory under diffeomorphisms i.e. we need to construct a *BMS invariant field theory*.

The first step in this quest consists on identifying the set of possible free fields compatible with such a huge symmetry group and theirs dynamic on $\Im$. This task can

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[2]From the point of view of the AdS/CFT correspondence, the boundary theory is invariant under the asymptotic symmetry group of an AdS spacetime. In the asymptotically flat scenario, the above introduced notion of diffeomorphism group of the boundary structure coincides with the asymptotic symmetry group first derived in [7].

be completed following the pattern that led Wigner to complete the same project for a Poincaré invariant theory living in Minkowski spacetime i.e. a free field is a wave function transforming under a unitary and irreducible representation (irrep.) of the Poincaré group. Thus, if we stick to such a definition, substituting Poincaré with BMS invariance, we need to classify and explicitly construct all the unitary irreps. of the Bondi-Metner-Sachs group. This programme has been completed in the late seventies by McCarthy both in the nuclear and in the Hilbert topology using Mackey's theory of induced representations and we will review here some important details (see [6, 8] and references therein for an exhaustive analysis).

Let us introduce the following concepts:

- the *character* of $N = C^\infty(S^2)$ which is a continuous group homomorphism $\chi : N \to U(1)$ which associates to each map χ a unique distribution β lying in N^*, the topological dual space of N, such that

$$\chi(\alpha) = e^{i(\alpha,\beta)}, \quad \forall \alpha \in N$$

where (β, α) stands for the evaluation of the distribution β with the suitable test function α[3],

- the *orbit* of a character χ i.e. the set

$$\mathcal{O}_\chi := \{g\chi \mid g = (\Lambda, \alpha) \in BMS\},$$

where $g\chi(\alpha) = \chi(g^{-1}\alpha)$ for any $\alpha \in N$.

- the *isotropy group* of χ:

$$H_\chi := \{g \in BMS \mid g\chi = \chi\}.$$

For the BMS group all the isotropy subgroups are $H_\chi = L_\chi \ltimes N$ where L_χ is a closed subgroup of $SL(2, \mathbb{C})$ called *little group*.

Thus, if we consider a character χ and a closed little group L_χ, it is possible to construct a unitary representation U of $L_\chi \ltimes N$ acting on a (non necessary finite dimensional) Hilbert space $\mathcal{H}$ as follows:

$$U(\Lambda, \alpha)\psi = \chi(\alpha)D(\Lambda)\psi, \tag{6}$$

where $D(\Lambda)$ is a unitary representation of L_χ. The key step consists on showing that (6) induces a unitary and irreducible representation of the BMS group which, thus, can be classified only by means of the possible little groups[4] L_χ. The latter problem has been studied in detail in [9] and the result consists on a plethora of subgroups of $SL(2, \mathbb{C})$, the most notables being the connected ones i.e. $SU(2)$, $SO(2)$ and Δ the double cover of two dimensional Euclidean group. These groups represent the key

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[3]We drop from now on the angular dependence of supertranslation restoring it if necessary to avoid confusion.

[4]This statement is not throughout complete since a key achievement consists on establishing if the list of all the above irreducible representation is complete. We leave a discussion of this problem to [6, 9].

ingredient to construct the kinematical configurations of a BMS invariant free field theory; we are now entitled to introduce the so-called *induced wave function* which is a map transforming under a unitary and irreducible representation of the full symmetry group i.e. a BMS free field

$$\psi : \mathcal{O}_\chi = \frac{SL(2,\mathbb{C})}{L_\chi} \to \tilde{\mathcal{H}}, \tag{7}$$

where $\tilde{\mathcal{H}}$ is a suitable target Hilbert space and where ψ transforms as

$$(\Lambda\psi)(p) = \sqrt{\frac{d\mu(\Lambda p)}{d\mu(p)}} D\left(\omega(p)^{-1}\Lambda\omega(\Lambda^{-1}p)\right)\psi(\Lambda^{-1}p), \; \Lambda \in SL(2,\mathbb{C}) \tag{8}$$

$$(\alpha\psi)(p) = p(\alpha)\psi(p), \quad \alpha \in C^\infty(S^2) \tag{9}$$

where p is a generic point on the orbit $\mathcal{O}_\chi = SL(2,\mathbb{C})\chi$ and, thus, it is a character. Furthermore $d\mu(p)$ is a suitably chosen measure on $\mathcal{O}_\chi$, ω is a global section of the bundle $\pi : SL(2,\mathbb{C}) \to \frac{SL(2,\mathbb{C})}{L_\chi}$ and $D(\Lambda)$ a unitary representation of L_χ.

The reader should notice that the induced wave function is defined in the space of characters though, as stated previously, to each χ we can associate a distribution over S^2 and consequently both (8) and (9) can be equivalently read as maps over N^*. Furthermore it is possible to associate to any of the above free fields a notion of mass with the following argument. Consider $N = C^\infty(S^2)$ whose elements can be seen as linear combinations of spherical harmonics $Y_{lm}(z,\bar{z})$. The set of the first four harmonics, i.e. $l = 0, 1$, span a four dimensional $SL(2,\mathbb{C})$-invariant normal subgroup of N isomorphic to T^4 (hence the name "supertranslations for $C^\infty(S^2)$). Consider now the set of distributions N^*, the topological dual space of N and the annihilator $T_0^4 = \left\{\beta \in N^* \mid (\alpha,\beta) = 0 \; \forall \alpha \in T^4 \subset C^\infty(S^2)\right\}$. It is possible to introduce the projection [9]

$$\pi : N^* \to \left(T^4\right)^* \sim \frac{N^*}{(T^4)^0},$$

where $\sim$ stands for a non canonical isomorphism and where, to each distribution, it is associated an element in the space generated by the dual spherical harmonics Y_{lm}^* implicitly defined as $(Y_{lm}^*, Y_{l'm'}) = \delta_{ll'}\delta_{mm'}$. Furthermore, since the above projection is $SL(2,\mathbb{C})$-invariant and since, by construction, $(T^4)^*$ is isomorphic to T^4 it is possible to associate to each $\beta \in N^*$ a scalar function over S^2 which is a linear combination of the first four harmonics. The coefficients of such a combination identify a 4-vector (also called the Poincaré momentum) which we indicate as $\pi[\beta]^\mu$ and consequently we can introduce the quantity

$$m^2 = \eta_{\mu\nu}\pi[\beta]^\mu\pi[\beta]^\nu,$$

which turns out to be a Casimir invariant (together with $sgn[\pi(\beta)_0]$) for the theory of <u>faithful</u> unitary and irreducible representations of the BMS group [9].

Consequently, in analogy with the Poincaré counterpart, it is natural to identify m^2 as the squared mass of a BMS free field and N^* as the space of supermomenta[5].

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[5]This is completely equivalent to Wigner's construction for a Poincaré invariant theory on Minkowski spacetime where T^4 played the role of coordinate space and $(T^4)^*$ that of momenta space.

Eventually, according to the analysis in [6, 9], we have all the ingredients for a full classification of the kinematical configurations of a BMS invariant free field. In particular, discarding negative values of m^2, the results for the connected little groups can be summarized in the following tabular:

Orbit	possible value for the mass
$\frac{SL(2,\mathbb{C})}{SU(2)}$	$m^2 > 0$
$\frac{SL(2,\mathbb{C})}{SO(2)}$	$m^2 \geq 0$
$\frac{SL(2,\mathbb{C})}{\Delta}$	$m^2 = 0$

At this stage, we are far from completing Wigner programme since we have also the chance to construct the dynamics of all the free fields only by means of the theory of representations. The starting point consists on noticing that, in physics, the concept of induced wave function is not commonly used whereas it is widely introduced the so-called *covariant wave function* i.e., in a BMS language,

$$\psi' : N^* \longrightarrow \mathcal{H}', \tag{10}$$

where $\mathcal{H}'$ is a suitable target Hilbert space and where ψ' transforms under a unitary but not necessarily irreducible representation of the BMS group i.e.

$$[U(g)\psi'](\beta) = \chi_\beta(\alpha)\tilde{D}(\Lambda)\psi'[\Lambda^{-1}\beta], \quad \forall g = (\Lambda, \alpha) \tag{11}$$

where $\tilde{D}(\Lambda)$ is a unitary $SL(2, \mathbb{C})$ representation and χ_β stands for the unique character associated to the β-distribution.

According to the previous discussion, it is immediate to realize that (11) does not represent a free field since the irreducibility of the representation is a request which cannot be easily given up. The rationale behind Wigner argument is that the covariant wave function can be made completely equivalent to (8) and to (9) if suitable constraints are imposed. Beside some technical details, fully accounted in [8], in a BMS framework, these constraints are threefold:

- an orbit equation which restricts the support of (11) from N^* to the finite dimensional orbit $\frac{SL(2,\mathbb{C})}{L_\chi}$:

$$[\beta - SL(2, \mathbb{C})\bar{\beta}]\psi'(\beta) = 0,$$

  where $SL(2, \mathbb{C})\bar{\beta}$ stands for the action of $SL(2, \mathbb{C})^6$ on the distribution $\bar{\beta}$ chosen in such a way that $L_\chi\bar{\beta} = \bar{\beta}$.

- a mass equation which associates to the ψ' a fixed value of m^2 i.e.

$$[\eta_{\mu\nu}\pi(\beta)^\mu\pi(\beta)^\nu - m^2]\psi'(\beta) = 0$$

- an equation which selects a unitary and irreducible representation $D(\Lambda)$ of a little group L_χ inside $\tilde{D}(\Lambda)$ defined in (11). This can be achieved introducing a suitable orthoprojector operator [8, 10] such that

$$\rho(\beta)\,\psi'(\beta) = \psi'(\beta).$$

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[6]For a generic element $\beta \in N^*$, $\Lambda\beta$ is defined in a distributional sense as $(\Lambda\beta, \alpha) = (\beta, \Lambda^{-1}\alpha)$ for any $\alpha \in N$.

The set of the above three equations represents in a momentum frame either the constraints to impose on a covariant wave function to be equivalent to an induced wave function either the equations of motion describing the dynamic of the associated free field. As a matter of fact, the same construction leads in a Poincaré invariant theory to the usual equations of Klein-Gordon, Dirac, Proca etc... Hence, we have recovered in a BMS framework the full set of kinematical and dynamical configurations for the free field theory and we summarize the results by means of the explicit example of a BMS massive real scalar field i.e. the induced wave function [11]:

$$\psi : \frac{SL(2, \mathbb{C})}{SU(2)} \longrightarrow \mathbb{R},$$

$$[g\psi](p) = p(\alpha)\psi(\Lambda^{-1}\beta). \quad g = (\Lambda, \alpha)$$

The above map is completely equivalent to its covariant counterpart

$$\psi' : N^* \longrightarrow \mathbb{R},$$

$$[U(\Lambda, \alpha)\psi'](\beta) = e^{i(\alpha,\beta)}\psi'(\Lambda^{-1}\beta), \tag{12}$$

supplemented with the BMS Klein-Gordon equations of motion (the orthoprojector $\rho(\beta) = 1$)

$$[\beta - \pi(\beta)]\psi'(\beta) = 0 \quad [\eta_{\mu\nu}\pi(\beta)^{\mu}\pi(\beta)^{\nu} - m^2]\psi'(\beta) = 0. \tag{13}$$

4 From boundary to bulk

The key question we need to address is whether the above (intrinsic) boundary data really encode the information from a fixed bulk theory. The main obstruction within this respect lies in the universality of the null infinity structure which is inherited by the BMS group i.e. a BMS field theory encodes a priori the data from all the asymptotically flat spacetimes.

In order to solve this problem, we face two candidate directions: the first consists on considering a fixed bulk and on introducing a suitable projection of the physical data on the boundary which are eventually interpreted as BMS invariant degrees of freedom. This line of reasoning has been first discussed in [6] and it will not be pursued here; conversely we will try to recover the bulk information starting only from boundary data. Following [11–13], we start from the following remark: the physical interpretation of Wigner construction strongly relies on a suitable identification of the support of the covariant wave function with a submanifold of the spacetime. In a Poincaré invariant theory, this is a straightforward consequence either of the isomorphism between T^4 and $\mathbb{R}^4$ either of the identification of $(T^4)^*$ with T^4 by means of the canonical pairing between vectors and co-vectors induced by the metric $\eta_{\mu\nu}$.

In a BMS invariant theory, it is clearly not reasonable to look for an identification of N^* with $\Im$ and, instead, we follow a slightly different path. It originates from an alternative formulation of general relativity (nonetheless fully equivalent to Einstein's

theory) which calls for dropping the metric as the fundamental field variable. This approach, also known as *null surface formulation* of general relativity, has been developed in the mid nineties (see [14] and references therein) and the starting point is a four dimensional asymptotically flat spacetime M, a generic point $x^a \in M$ together with $L(x^a, x'^a) = 0$, the light cone equation connecting x_a to x'_a. If we consider $x'^a \in \Im^+$ and if we introduce the Bondi frame, then the boundary point x'^a can be parametrized as (u, θ, φ); furthermore the equation $L(x^a, x'^a) = 0$ now depends on the u-variable and it can be inverted as

$$L(x^a, u, \theta, \varphi) = 0 \longrightarrow u = Z(x^a, \theta, \varphi). \tag{14}$$

The inverse of L is not a priori unique and the set of Z functions, also known as "cut functions", represents the fundamental variable in the null surface formulation of Einstein theory. Thus, within this respect, the metric now becomes a functional dependent on the set of scalar maps over $M \times S^2$ defined in (14); the full consistency of this approach and the derivation of Einstein equations has been discussed in detail by several authors (still refer to [14] and references therein) and we will not review their results here. Conversely we concentrate on the interpretation of the cut function namely, if we held fixed the boundary point on $\Im$, (14) represents the past light rays originating from (u, θ, φ). More importantly, if we held fixed the bulk point, (14) describes the intersection of the light cone originating from x_a and intersecting $\Im$ on a 2-surface which turns out to be homotopically equivalent to S^2.

Up to now we have discarded the chance that the light rays emanating from x^a intersect at a certain point thus originating a caustic. Even if this pathology should be cured in detail, to our purposes, it suffice to notice, that it always exists a suitable neighbourhood of $\Im$ where Z is a unique and differentiable scalar function i.e. $Z_{x^a}(\theta, \varphi) \in C^\infty(S^2)$. Consequently we may interpret each Z_{x^a} as a BMS supertranslation and, furthermore, these supertranslations reconstruct, up to the conformal factor needed to compactify the asymptotically flat spacetime, the full geometry of the bulk.

It is a natural suspicion that the reconstruction of the bulk data by means of cut functions/supertranslations can be transported from the geometrical setting to the field theoretical framework and we conjecture that the information of a fixed asymptotically flat bulk spacetime M is encoded in the BMS fields whose support is compatible with the supertranslations[7] reconstructing M in the null surface formalism.

Though we cannot supplement the above statement with a complete proof, we provide here a concrete example reconstructing the 2-point function for the massive scalar field in Minkowski spacetime starting from the BMS counterpart. Let us thus start from (12) and (13) which are written in a supermomentum frame and let us transform them in a supertranslation frame. This task can be achieved by means of the infinite dimensional counterpart of the Fourier transform discussed in [11] and defined over the Hilbert space of functions over a second suitably chosen Hilbert space. Two ingredients are needed:

- since N and N^* form the Gelfand triplet $N \subset L^2(S^2) \subset N^*$, we first need to notice that the support equation (13) grants us that the massive BMS scalar field

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[7]Though the support of the wave functions is N^* in a momentum frame, there is no contradiction in our hypothesis if we remember that N and N^* form the Gelfand triplet $N \subset L^2(S^2) \subset N^*$.

lives on a mass hyperboloid generated by the $SL(2, \mathbb{C})$ action over the $SU(2)$-fixed point in N^*. According to the analysis in [9], such a point is actually a smooth function over S^2 and thus the entire orbit lies in $C^\infty(S^2)$ and consequently in $L^2(S^2)$. Thus, if we require that each ψ, as in (12), lies in the space $\mathcal{H}'' = L^2(L^2(S^2), \mu)$ where μ is the unique Gaussian measure associated to $L^2(S^2)$, we are entitled to use the infinite dimensional Fourier transform theory.

- we can define two important operators acting on an element ψ lying in $\mathcal{H}''$ namely the *multiplication operator* along the η-direction in $L^2(S^2)$:

$$Q_\eta \psi(x) = < \eta, x > \psi(x),$$

where $<, >$ is the internal product over $L^2(S^2)$ and the *derivative* operator along the η-direction:

$$\mathcal{D}_\eta \psi(x) = \lim_{t \to 0} \frac{\psi(x + t\eta) - \psi(x)}{t}.$$

Both operators are related by means of the Fourier transform $\mathcal{F}$ as $\mathcal{F}Q_\eta = i\mathcal{D}_\eta\mathcal{F}$ and viceversa.

Bearing in mind these remarks, we can write (13) in terms of multiplication and derivative operators as

$$Q_{Y_{lm}}\psi(x) = 0, \quad [\eta^{\mu\nu}Q_{e_\mu}Q_{e_\nu} - m^2]\psi(x) = 0,$$

where $l > 1$ and $e_\mu = \{Y_{00}, ..., Y_{11}\}$. Acting with the Fourier transform, we end up with:

$$\mathcal{D}_{Y_{lm}}\psi(x) = 0, \quad [\eta^{\mu\nu}\mathcal{D}_{e_\mu}\mathcal{D}_{e_\nu} - m^2]\psi(x) = 0. \tag{15}$$

These can be read as the Euler-Lagrange equations for

$$L[\psi, \gamma_{lm}] = \psi(x)[\eta^{\mu\nu}\mathcal{D}_{e_\mu}\mathcal{D}_{e_\nu} - m^2]\psi(x) + \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \gamma_{lm}(x)\mathcal{D}_{Y_{lm}}\psi(x), \tag{16}$$

where each $\gamma_{lm}(x)$ is a suitable Lagrange multiplier. We can now write the action for a BMS Klein-Gordon field integrating the Lagrangian over the above introduced Gaussian measure $d\mu(x)$. Eventually we can formulate a path-integral as

$$Z[\psi, \gamma_{lm}] = \int_{\mathcal{C}} d\mu[\psi, \gamma_{lm}]e^{iS[\psi, \gamma_{lm}]}, \quad S[\psi, \gamma_{lm}] = \int_{L^2(S^2)} d\mu[x]L[\psi, \gamma_{lm}],$$

where $\mathcal{C}$ is a set of suitably chosen kinematical configurations. Z reduces to the following expression [11]:

$$Z[\psi, \gamma_{lm}] = \int_{\mathcal{C}} d\mu[\psi]e^{i<\phi(x)B\phi(x)>},$$

10

where $<,>$ is the internal product on $L^2(S^2)$ and

$$B = \eta^{\mu\nu}\mathcal{D}_{e_\mu}\mathcal{D}_{e_\nu} - m^2 + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{1}{2\zeta_{lm}}(Q_{Y_{lm}} - \mathcal{D}_{Y_{lm}})\mathcal{D}_{Y_{lm}}, \qquad (17)$$

where ζ_{lm} are arbitrary real number. Whenever the partition function assumes the above expression, the Feynman propagator $D_F(x_1 - x_2)$ satisfies the equation

$$BD_F(x_1 - x_2) = i\delta(x_1 - x_2),$$

which, upon Fourier transform and using the definition of multiplication operator, becomes the differential equation

$$[\eta^{\mu\nu}k_\mu k_\nu - m^2 + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{1}{2\zeta_{lm}}(k_{lm} - \mathcal{D}_{Y_{lm}})k_{lm}]D_F(k) = i, \qquad (18)$$

where $k_\mu = <e_\mu, k>$ and $k_{lm} = <Y_{lm}, k>$. We can now appeal to our conjecture and we look for those supertranslation/cut functions reconstructing Minkowski spacetime. This is rather simple example from the point of view of the null surface formulation since the light cone equation can be globally inverted yielding $Z(x^\mu, \theta, \varphi) = x^\mu e_\mu$ where, as before, e_μ are only the first four spherical harmonics. Similarly in a momentum frame we end up with $Z(k^\mu, \theta, \varphi) = k^\mu e_\mu$ which, substituted in (18), gives

$$[\eta^{\mu\nu}k_\mu k_\nu - m^2]D_F(k) = i \Longrightarrow D_F(k) = \frac{i}{\eta^{\mu\nu}k_\mu k_\nu - m^2}, \qquad (19)$$

which is exactly the Feynman propagator for a massive real scalar field in Minkowski spacetime.

5   Conclusions

We have discussed the kinematical and dynamical data of a free field theory living on the null infinity boundary of an asymptotically flat spacetime and representing the candidate to holographically encode the bulk information. We have also proposed a concrete way to reconstruct such information by means of the null surface formulation of general relativity which allows us to interpret the BMS supertranslations as the degrees of freedom encoding the geometrical informations from the bulk. Starting from this rationale, we have recovered the Feynman propagator for a Minkowski massive real scalar field using only the data from the boundary counterpart. Though this is only a simple example, we believe that it provides a good signal that holography can be implemented successfully on an asymptotically flat manifold and that the BMS group should play a central role. To conclude we wish to stress that the null nature of $\Im$ and the intrinsic difficulty to work with an infinite dimensional group, such as the BMS, naturally leads to treat holography in such a scenario with the rigorous mathematical tools proper of the algebraic formulation of quantum field theory. In particular we hope to provide rigorous holographic result similar to those discussed by Rehren in [15, 16]

in the framework of AdS/CFT. A preliminary analysis in [6] indicates that, at least in the massless case, holography could be implemented in terms of Weyl algebra and the extension of these results to massive fields represents the next challenge.

Acknowledgments

References

[1] R. M. Wald, *" General Relativity"* Chicago Univ. Press (1984)

[2] G. 't Hooft, arXiv:gr-qc/9310026,

[3] R. Bousso, Rev. Mod. Phys. **74** (2002) 825

[4] R. Bousso, E. E. Flanagan and D. Marolf, Phys. Rev. D **68** (2003) 064001

[5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323** (2000) 183

[6] C. Dappiaggi, V. Moretti and N. Pinamonti, arXiv:gr-qc/0506069.

[7] H. Bondi, M. G. J. van der Burg and A. W. K. Metner, Proc. Roy. Soc. Lond. A **269** (1962) 21.

[8] G. Arcioni and C. Dappiaggi, Nucl. Phys. B **674** (2003) 553

[9] P.J. McCarthy: Proc. R. Soc. London **A343** (1975) 489,

[10] A. O. Barut, R. Raczka: *"Theory of group representations and applications"* World Scientific (1986),

[11] C. Dappiaggi, JHEP **0411** (2004) 011

[12] G. Arcioni and C. Dappiaggi, Class. Quant. Grav. **21** (2004) 5655

[13] C. Dappiaggi, Phys. Lett. B **615** (2005) 291

[14] S. Frittelli, C. Kozameh, E. T. Newman, J. Math. Phys. **36** (1995) 4984,

[15] K. H. Rehren, Annales Henri Poincare **1** (2000) 607,

[16] K. H. Rehren Phys. Lett. B **493** (2000) 383,